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Density of the Linear Span of Principal Solutions of Second Order Linear Differential Equations

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INTRODUCTION

By a change of variable, the classical Weierstrass approximation theorem can be stated as follows.

THEOREM (Weierstrass). *Suppose f is a real valued continuous function on $[0, \infty)$, $f(x)\, e^{\mu x}$ $(\mu \geqslant 0)$ has a finite limit at $x = \infty$ and $\varepsilon > 0$. Then there correspond real numbers $c_0, \ldots, c_n$ such that*

$$\left| f(x) - \sum_{j=0}^{n} c_j e^{-(\mu + j)x} \right| < \varepsilon e^{-\mu x}, \qquad x \geqslant 0.$$

Our main result, Theorem II (see Section 2), asserts that an analogue of the above version of the Weierstrass theorem is obtained when the functions $e^{-\mu x}$ are replaced by the principal solutions $e_\mu(x)$. The function $e_\mu(x)$, for $\mu > Q^{1/2}$, is defined as the solution of

$$u'' = (q(x) + \mu^2)u, \quad u(0) = 1, \quad u(\infty) = 0 \qquad \left(u'(x) = \frac{d}{dx}\, u(x)\right).$$

Principal solutions are discussed in [1, p. 355]. We assume that $q(x)$ is a real valued continuous function on $0 \leqslant x < \infty$ and that $\sup |q(x)| = Q < \infty$. The existence of e_μ is guaranteed by Lemma 1.1. More precisely, our density theorem asserts that an approximation

$$\left| f(x) - \sum_{j=0}^{n} c_j e_{\lambda + j}(x) \right| < \varepsilon e^{-\mu x}, \qquad x \geqslant 0,$$

where $\lambda = (\mu^2 + Q)^{1/2}$, is possible for each real valued continuous f on $[0, \infty)$ which is $o(e^{-\mu x})$ as $x \to \infty$. We know of no other papers that investigate the

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density of the linear span of a class of principal solutions. The question of completeness of eigenfunctions of Sturm–Liouville problems has been extensively investigated, but this question is only remotely related to our problem.

As a major step in the proof of the density theorem, we obtain Theorem I, which is a representation theorem for e_μ; it asserts that e_μ can be represented in the form

$$\gamma_\mu e_\mu(x) = e^{-\mu x} + \int_x^\infty K(x, s)\, e^{-\mu s}\, ds$$

for a certain K which depends on q but not on μ (γ_μ is a positive constant). Marcenko [2, Lemma 1.1.1] obtains a similar representation in a different setting by clever use of an identity that frees K from the parameter μ. The use of an analogous identity is a major step in our proof.

1. The Representation Theorem

In this section we establish the existence and elementary properties of the principal solutions e_μ in Lemma 1.1 and we prove the representation theorem for e_μ, Theorem I.

Lemma 1.1. *Let $q(x)$ be a real valued continuous function on $0 \leqslant x < \infty$ such that $Q = \sup |q(x)|$ $(x \geqslant 0)$ is finite. For each $\mu > Q^{1/2}$, the boundary value problem*

$$u'' = (q + \mu^2)u, \qquad u(0) = 1, \qquad u(\infty) = 0$$

has a unique solution, which we denote by $e_\mu(x)$, $0 \leqslant x < \infty$. Furthermore, (i) $e_\mu(x) > 0$, (ii) $e'_\mu(x) \leqslant 0$, *and* (iii) $e_\mu(x) = O(e^{-rx})$ *as $x \to \infty$ for each $0 < r < (\mu^2 - Q)^{1/2}$.*

The proof of the lemma will be based on the following two lemmas.

Lemma 1.2. *Suppose $P(x)$ is a real valued continuous function on $0 \leqslant x < \infty$ such that $P(x) \geqslant 0$. If $u'' = Pu$, $u(0) = 1$, $u'(0) = 1$, then* (i) $u'(x) \geqslant 1$, (ii) $u(x) \geqslant 1 + x$, (iii) $\int_0^\infty (1/u^2(x))\, dx < \infty$ *and* (iv) $u'(x)$ *is nondecreasing.*

Lemma 1.3. *Let P, u be as in Lemma 1.2 and let $u_0(x) = u(x) \int_x^\infty (1/u^2(t))\, dt$. Then* (i) u_0 *is a solution of $u'' = Pu$,* (ii) $u_0(x) > 0$, (iii) $u_0'(x) \leqslant 0$ *and* (iv) $u_0(x) \leqslant (1 + x)/u(x)$.

For Lemma 1.3, (i)–(iii) are easily verified. For (iv), first note that $u(x) -$

$u(0) = \int_0^x u'(t)\,dt \leqslant xu'(x)$, since u' is nondecreasing. Thus, $u_0(x) \leqslant (u(0) + xu'(x)) \int_x^\infty (1/u^2)\,dt \leqslant (1+x)\,u'(x) \int_x^\infty (1/u^2)\,dt \leqslant (1+x) \int_x^\infty (u'/u^2)\,dt = (1+x)/u(x)$.

Now to prove Lemma 1.1 we let u and u_0 be defined as above for $P = q + \mu^2$. We define $e_\mu(x)$ as $(u_0(0))^{-1} u_0(x)$. It is easily shown that for each $0 < r < (\mu^2 - Q)^{1/2}$, $u(x) \geqslant (1+r)\,e^{rx}/2r + (r-1)\,e^{-rx}/2r$, since $q + \mu^2 > r^2$. The lemma now follows from Lemma 1.3.

THEOREM I. *Let q, Q and e_μ be as in Lemma* 1.1. *There exists a function $K(x, s)$ which satisfies the following conditions.*

(1) *$K(x, s)$ is real valued and continuous on $0 \leqslant x, s < \infty$.*

(2) *$K(x, s) = 0$ for $x > s$.*

(3) *For $\mu > Q^{1/2}$ and u continuous and $O(e^{-\mu x})$:* (i) *the integral $v(x) = \int_x^\infty K(x, s)\,u(s)\,ds$ converges absolutely for each $x \geqslant 0$,* (ii) *v is continuous,* (iii) $\sup |v(x)|\, e^{\alpha x} \leqslant C \sup_{x \geqslant 0} |u(x)|\, e^{\mu x}$, *where $\alpha = (\mu^2 - Q)^{1/2}$ and $C = (1 - [Q/2\mu(\mu + (\mu^2 - Q)^{1/2})])^{-1}$.*

(4) *For $\mu > Q^{1/2}$,*

$$\gamma_\mu e_\mu(x) = e^{-\mu x} + \int_x^\infty K(x, s)\,e^{-\mu s}\,ds,$$

where γ_μ is a nonzero constant which depends on μ.

If we combine (iii) and (iv) of the theorem, we get the following corollary, which strengthens (iii) of Lemma 1.1.

COROLLARY 1.1. *For $\mu > Q^{1/2}$, $e_\mu(x)$ is $O(e^{-\alpha x})$ as $x \to \infty$, where $\alpha = (\mu^2 - Q)^{1/2}$.*

Proof of Theorem I. For any complex number μ, the Volterra integral equation

$$u(x) = ae^{-\mu x} + be^{\mu x} + \int_0^x (2\mu)^{-1}(e^{\mu(x-t)} - e^{-\mu(x-t)})\,q(t)\,u(t)\,dt$$

has a unique solution which is also the unique solution of the initial value problem

$$u'' = (q + \mu^2)u, \qquad u(0) = a + b, \qquad u'(0) = -\mu a + \mu b.$$

From now on we let $\mu > Q^{1/2}$. Thus,

$$e_\mu(x) = a_\mu e^{-\mu x} + b_\mu e^{\mu x} + \int_0^x (2\mu)^{-1}(e^{\mu(x-t)} - e^{-\mu(x-t)})\,q(t)\,e_\mu(t)\,dt, \quad (1.1)$$

where

$$a_\mu + b_\mu = 1,$$
$$-\mu a_\mu + \mu b_\mu = e'_\mu(0).$$

Since the integral $\int_0^\infty e^{-\mu t} q(t)\, e_\mu(t)\, dt$ converges absolutely, we can rewrite Eq. (1.1) as

$$e_\mu(x) = a_\mu e^{-\mu x} + e^{\mu x}\left(b_\mu + (2\mu)^{-1}\int_0^\infty e^{-\mu t} q(t)\, e_\mu(t)\, dt\right)$$
$$-(2\mu)^{-1}\int_0^\infty e^{-\mu|x-t|} q(t)\, e_\mu(t)\, dt. \tag{1.2}$$

The term in Eq. (1.2) that is a constant multiple of $e^{\mu x}$ must be 0 since all other terms in the equation tend to 0 as x tends to ∞. Thus,

$$\begin{aligned} a_\mu &= 1 - b_\mu, \\ b_\mu &= -\int_0^\infty (2\mu)^{-1} e^{-\mu t} q(t)\, e_\mu(t)\, dt, \\ e_\mu(x) &= a_\mu e^{-\mu x} - \int_0^\infty (2\mu)^{-1} e^{-\mu|x-t|} q(t)\, e_\mu(t)\, dt. \end{aligned} \tag{1.3}$$

Let K_μ denote the kernel

$$K_\mu(x, t) = (2\mu)^{-1} e^{-\mu|x-t|} q(t), \qquad 0 \leqslant x, \quad t < \infty,$$

$\tilde{K}_\mu$ the corresponding operator on functions, and $\|\tilde{K}_\mu\|$ the operator norm for the function space $L_1[0, \infty)$. It is known that for a general continuous kernel K,

$$\|\tilde{K}\| = \sup_s \int_0^\infty |K(x, s)|\, dx \qquad (s \geqslant 0).$$

From this, Eq. (1.3) and the fact that $|e_\mu(x)| \leqslant 1$, we conclude that

$$\begin{aligned} &\|\tilde{K}_\mu\| \leqslant Q/\mu^2 < 1, \qquad a_\mu \geqslant 1 - Q/2\mu^2 \geqslant \tfrac{1}{2} \\ &(a_\mu)^{-1} e_\mu(x) = e^{-\mu x} + \sum_{n=1}^\infty (-1)^n (\tilde{K}_\mu)^n e^{-\mu x}, \qquad \mu > Q^{1/2}, \end{aligned} \tag{1.4}$$

where the series converges in the form of $L_1[0, \infty)$.

We will now obtain the kernel K of the theorem by showing that

$(\tilde{K}_\mu)^n e^{-\mu x} = \int_x^\infty K_n(x, s)\, e^{-\mu s}\, ds$ for some K_n (independent of μ) and then defining K as $\sum (-1)^n K_n$ $(n = 1, 2,\ldots)$. For $n = 1, 2,\ldots,$

$$(\tilde{K}_\mu)^n e^{-\mu x} = (2\mu)^{-n} \int e^{-\mu|x-t_n|} q(t_n)\, e^{-\mu|t_n - t_{n-1}|} q(t_{n-1}) \cdots$$
$$\times e^{-\mu|t_2 - t_1|} q(t_1)\, e^{-\mu t_1}\, dt_1 \cdots dt_n, \tag{1.5}$$

where the integration is over the set $t_j \geqslant 0$, $j = 1,\ldots, n$. If we use the formula

$$\mu^{-n} e^{-\mu h} = \Gamma(n)^{-1} \int_h^\infty (s-h)^{n-1} e^{-\mu s}\, ds, \qquad h \geqslant 0, \quad n = 1, 2,\ldots,$$

then the right side of Eq. (1.5) becomes

$$2^{-n} \int \left[\Gamma(n)^{-1} \int_{s \geqslant |x - t_n| + \cdots + t_1} (s - (|x - t_n| + \cdots + t_1))^{n-1} e^{-\mu s}\, ds \right]$$
$$\times q(t_1) \cdots q(t_n)\, dt_1 \cdots dt_n. \tag{1.6}$$

We define $K_n(x, s)$ as the kernel obtained by formally interchanging the order of integration in (1.6); that is, we define K_n as follows. For $x \leqslant s$,

$$K_n(x, s) = (2^n \Gamma(n))^{-1} \int_{T_{x,s}} [s - (|x - t_n| + \cdots + t_1)]^{n-1}$$
$$\times q(t_1) \cdots q(t_n)\, dt_1 \cdots dt_n,$$

where $T_{x,s}$ is the set of n-tuples $(t_1,\ldots, t_n)$ such that (i) $s \geqslant |x - t_n| + |t_n - t_{n-1}| + \cdots + t_1$ and (ii) $t_1 \geqslant 0,\ldots, t_n \geqslant 0$; and, for $x > s$, $K_n(x, s) = 0$.

We need the following properties of K_n.

LEMMA 1.4. (i) *$K_n(x, s)$ is continuous.*

(ii) *$K_n(x, s) = 0$ for $x > s$.*

(iii) $(\tilde{K}_\mu)^n e^{-\mu x} = \int_x^\infty K_n(x, s)\, e^{-\mu s}\, ds.$

Properties (i) and (ii) are clear from the definition of K_n. To prove (iii) we note that since the integral in Eq. (1.5) is absolutely convergent and the integrand of the inner integral in (1.6) is nonnegative, the integral in (1.6) is also absolutely convergent. Thus, the order of integration in (1.6) can be interchanged. This proves (iii).

The crude bound

$$|K_n(x, s)| \leqslant (2^n \Gamma(n))^{-1} Q^n (s - x)^{n-1} (2s)^n, \qquad n = 1, 2,\ldots \tag{1.7}$$

follows from the definition of K_n when we note that for $(t_1, \ldots, t_n)$ in $T_{x,s}$, $s \geqslant |x - t_j|$ for $j = 1, 2, \ldots, n$. Thus, the series

$$\sum_{n=1}^{\infty} (-1)^n K_n(x, s) \tag{1.8}$$

converges absolutely. We define $K(x, s)$ as the sum of this series. The bound (1.7) also shows that the series converges locally uniformly; this, together with (i) and (ii) of Lemma 1.4, establishes (1) and (2) of Theorem I. For (i) and (ii) of (3), first consider the series

$$\sum_{n=1}^{\infty} (-1)^n \int_x^{\infty} K_n(x, s)\, u(s)\, ds. \tag{1.9}$$

For $M \geqslant x$, the crude bound (1.7) shows that

$$\int_M^{\infty} |K_n(x, s)\, u(s)|\, ds \leqslant C_n \int_M^{\infty} s^{2n-1} e^{-\mu s}\, ds.$$

This estimate shows that the integral

$$\int_x^{\infty} K_n(x, s)\, u(s)\, ds \tag{1.10}$$

converges absolutely and uniformly for x in any finite interval as the upper limit of the integral $\to \infty$. This and the fact that K_n is continuous shows that (1.10) is continuous. Let $K_{Q,n}$ denote K_n in the case $q \equiv Q$ (q is fixed throughout this proof). Clearly, $|K_n(x, s)| \leqslant K_{Q,n}(x, s)$. If we apply Lemma 1.4(iii) to $K_{Q,n}$ and then bound the left hand side of Lemma 1.4(iii) by estimating the integral (corresponding to $q \equiv Q$) in Eq. (1.5) as an iterated integral, we obtain

$$\int_x^{\infty} K_{Q,n}(x, s)\, e^{-\mu s}\, ds \leqslant (Q/\mu^2)^n.$$

Thus, the absolute value of the nth term in the series (1.9) is dominated by $(Q/\mu^2)^n \|u\|_\mu$, where, for convenience in the remainder of the proof, $\|u\|_\mu$ denotes $\sup |u(x)\, e^{\mu x}|$ $(x \geqslant 0)$ for any real valued continuous function u on $[0, \infty)$. Since $Q/\mu^2 < 1$, the series (1.9) converges uniformly; and, since the nth term of the series is continuous, the sum of the series is continuous. Since

$$\left| \left(\sum_{n=1}^{m} (-1)^n K_n(x, s) \right) u(s) \right| \leqslant \sum_{n=1}^{\infty} K_{Q,n}(x, s)\, |u(s)| \qquad \text{for} \quad m = 1, 2, \ldots$$

and the latter has a finite integral over $[x, \infty)$ by what we showed above, together with the Montone Convergence theorem, we conclude from the Dominated Convergence theorem that the series in (1.9) is $v(x) = \int_x^\infty K(x, s) u(s)\, ds$. We have now proved parts (i) and (ii) of (3).

Note that the estimates above only show that $|v(x)| \leqslant (\sum (Q/\mu^2)^n) \|u\|_\mu$, which is not good enough for (iii) of (3). We will now prove (iii). Let H_n and H_μ denote K_n and K_μ, respectively, for the case $q \equiv -Q$. If we note that $|K_n(x, s)| \leqslant (-1)^n H_n(x, s)$ and apply Lemma 1.4(iii), we obtain the following inequalities:

$$\begin{aligned}\int_x^\infty |K(x, s)|\, |u(s)|\, ds &\leqslant \sum_{n=1}^\infty \int_x^\infty |K_n(x, s)|\, |u(s)|\, ds \\ &\leqslant \left(\sum_{n=1}^\infty (-1)^n \int_x^\infty H_n(x, s)\, e^{-\mu s}\, ds \right) \|u\|_\mu \qquad (1.11) \\ &\leqslant \left(e^{-\mu x} + \sum_{n=1}^\infty (-1)^n (\tilde{H}_n)^n e^{-\mu x} \right) \|u\|_\mu .\end{aligned}$$

From (1.4), using the case $q \equiv -Q$, we have

$$e^{-\mu x} + \sum_{n=1}^\infty (-1)^n (\tilde{H}_\mu)^n e^{-\mu x} = (a_\mu)^{-1} e^{-\alpha x}, \tag{1.12}$$

where the series converges and the identity holds in the norm of $L_1[0, \infty)$ (a_μ corresponds to $q \equiv -Q$). However, since $-\tilde{H}_\mu$ has a positive kernel, so does $(-\tilde{H}_\mu)^n$. Thus, the series in (1.12) is pointwise increasing. We can now conclude that (1.12) holds a.e. Combining (1.11) and (1.12), we obtain the inequality in (3)(iii) of Theorem I. Using the formula (1.3) for a_μ we obtain

$$a_\mu = 1 - [Q/2\mu(\mu + (\mu^2 - Q)^{1/2})].$$

We have proved (iii).

To prove (4) we first note that in (1.4) $a_\mu \neq 0$ and that (1.4) holds as an identity in the space $L_1[0, \infty)$. However, we know that e_μ is continuous on $[0, \infty)$. We also know that

$$(\tilde{K}_\mu)^n e^{-\mu x} = \int_x^\infty K_n(x, s)\, e^{-\mu s}\, ds, \tag{1.13}$$

the integral of the right hand side of (1.13) is continuous (the proof follows (1.10)), and the series on the right hand side of (1.4) converges uniformly (because of the inequality following (1.10)). Thus, we conclude that (1.4) holds as an equality between numbers for each x. To complete the proof of (4) it suffices to show that

$$e^{-\mu x} + \sum_{n=1}^{\infty} (-1)^n \int_x^{\infty} K_n(x, s)\, e^{-\mu s}\, ds = (I + \tilde{K})\, e^{-\mu x} \qquad (1.14)$$

for each $x \geqslant 0$. For a given x, we know that the sequence

$$\sum_{n=1}^{m} (-1)^n K_n(x, s)\, e^{-\mu s}, \qquad m = 1, 2, \ldots, \qquad (1.15)$$

converges pointwise to $K(x, s)\, e^{-\mu s}$ by the definition of K. However, the inequality following (1.10) shows that the sequence (1.15) has a dominating function in $L_1[x, \infty)$. Thus, (1.14) holds for each x by the Lebesgue Dominated Convergence theorem.

2. The Density Theorem

In this section we prove the following density theorem.

THEOREM II. *Let q, Q and e_μ be as in Lemma 1.1. Given that $\mu > Q^{1/2}$, f is a real valued continuous function on $[0, \infty)$, $f(x) = o(e^{-\mu x})$ as $x \to \infty$, and $\varepsilon > 0$, there correspond real numbers $c_0, c_1, \ldots, c_n$ such that*

$$\left| f(x) - \sum_{j=0}^{n} c_j e_{\lambda+j}(x) \right| < \varepsilon e^{-\mu x}, \qquad x \geqslant 0,$$

where $\lambda = (\mu^2 + Q)^{1/2}$.

Note. We have already pointed out in Corollary 1.1 that each $e_{\lambda+j}$ is $O(e^{-\mu x})$.

Proof. We assume throughout that $\mu > Q^{1/2}$. For each pair of real numbers (a, b), $0 \leqslant a < b < \infty$, we define a corresponding function w as follows: $w(t) = 1$, $0 \leqslant t \leqslant a$; $w(t) = 0$, $t \geqslant b$; on $a \leqslant t \leqslant b$, $w(t)$ is linear and $w(a) = 1$, $w(b) = 0$. Let K correspond to q as in Theorem I. We define the kernels K_w by: $K_w(x, t) = K(x, t)\, w(t)$. In the first part of the proof w is arbitrary; a particular choice of w will be made later. For convenience, we will let E_μ $(\mu \geqslant 0)$ denote the space of real valued continuous functions $f(x)$ on $0 \leqslant x < \infty$ for which

$$\sup |f(x)|\, e^{\mu x} \qquad (x \geqslant 0)$$

is finite and we let $\|f\|_\mu$ denote this sup. For any kernel function $K(x, s)$ we let $\tilde{K}$ denote a corresponding operator on a function space.

Before continuing with the proof of the theorem, we will establish three lemmas.

LEMMA 2.1. *$\tilde{K}_w$ is a bounded linear operator on E_μ.*

LEMMA 2.2. *As an operator on E_μ, $I + \tilde{K}_w$ has a bounded inverse.*

To prove Lemma 2.1, take $\mu > Q^{1/2}$ and $f \in E_\mu$. Clearly $\tilde{K}_w f$ is a continuous real valued function and $\tilde{K}_w f = \tilde{K}(wf)$. Let $\lambda = (\mu^2 + Q)^{1/2}$. Note that $\lambda > \mu$. For $x \leqslant b$,

$$|w(x) f(x)| \leqslant \|f\|_\mu e^{-\mu x} \leqslant \|f\|_\mu e^{-\lambda x} e^{(\lambda-\mu)b};$$

and, for $x > b$, $w(x) f(x) = 0$. Thus,

$$\|wf\|_\lambda \leqslant e^{(\lambda-\mu)b} \|f\|_\mu .$$

Hence,

$$\|\tilde{K}_w f\|_\mu = \|\tilde{K} wf\|_\mu \leqslant C \|wf\|_\lambda \leqslant C e^{(\lambda-\mu)b} \|f\|_\mu ,$$

where C corresponds to λ and Q as in Theorem I. Since $\tilde{K}$ is clearly linear, we have proved Lemma 2.1.

To prove Lemma 2.2, let $f \in E_\mu$. Using the usual estimates for Volterra type integral operators, we obtain

$$|(\tilde{K}_w)^n f(x)| \leqslant (1/n!)\, M^n (b-x)^n \|f\|_\mu , \qquad 0 \leqslant x \leqslant b,$$

where $M = \sup |K(x, t)|$ $(0 \leqslant x \leqslant t \leqslant b)$. In particular,

$$|(\tilde{K}_w)^n f(x)| \leqslant (1/n!)(Mb)^n \|f\|_\mu e^{\mu b} e^{-\mu x}.$$

Hence, corresponding to the function space E_μ, the operator norm

$$\|(\tilde{K}_w)^n\| \leqslant (1/n!)(Mb)^n e^{\mu b}.$$

This shows that the Neumann expansion $\sum_{n=0}^{\infty} (-\tilde{K}_w)^n$ converges in the operator norm; and, hence, it converges to the inverse of $I + \tilde{K}$. So Lemma 2.2 is proved.

LEMMA 2.3. *Given that f is a real valued continuous function on $[0, \infty)$, $f(x) = o(e^{-\mu x})$ as $x \to \infty$, and $\varepsilon > 0$, there corresponds a w (that is, a choice of a, b) such that $\|(1 - w) g\|_\mu < \varepsilon$, where g is the solution of $f = (I + \tilde{K}_w) g$.*

Proof. We first note that $(1 - w) g = 0$ for $0 \leqslant x \leqslant a$. Choose $r > 0$ so that $|f(x)| < \varepsilon e^{-\mu x}$, $x \geqslant r$. This and the fact that $(1-w)g = f$ for $b \leqslant x$ show that

$$|(1-w)g| < \varepsilon e^{-\mu x}, \qquad x \geqslant \max(b, r).$$

We must now estimate $(1-w)g$ for $a \leqslant x \leqslant b$. But,

$$g(x) = f(x) + \sum_{n=1}^{\infty} (-\tilde{K}_w)^n f(x).$$

Using the same argument as above, we obtain

$$|(\tilde{K}_w)^n f(x)| \leqslant (1/n!)\, M^n (b-x)^n B, \qquad a \leqslant x \leqslant b,$$

where $B = \sup |f(x)|$ $(a \leqslant x \leqslant b)$ and M is the same as above. Thus,

$$\sum_{n=1}^{\infty} |(-\tilde{K}_w)^n f(x)| \leqslant e^{M(b-a)} \varepsilon e^{-\mu a}$$

$$\leqslant e^{M(b-a)} \varepsilon e^{\mu(b-a)} e^{-\mu x}, \qquad \text{if} \quad r \leqslant a \leqslant x \leqslant b.$$

If we first choose $b > r$, then choose a so that $r \leqslant a$ and $(M+\mu)(b-a) \leqslant 1$ (recall that M depends on b but not a), then combine the above estimates we obtain

$$|(1-w)g| < (1+e)\, \varepsilon e^{-\mu x}, \qquad x \geqslant 0.$$

This completes the proof.

We now continue with the proof of Theorem II. Suppose f, μ and ε satisfy the hypothesis. Let $\lambda = (\mu^2 + Q)^{1/2}$. For any choice of w, there is a g in E_μ such that

$$f = g + \tilde{K}_w g = (1-w)g + (I + \tilde{K})\, wg. \tag{2.1}$$

By Lemma 2.3 we can choose w so that

$$\|(1-w)g\|_\mu < \varepsilon. \tag{2.2}$$

Since $wg = o(e^{-\lambda x})$ as $x \to \infty$, by the Weierstrass theorem, there are real numbers $b_0, b_1, \ldots, b_n$ such that

$$\|wg - h\|_\lambda < \varepsilon,$$

where $h(x) = \sum b_j e^{-(\lambda + j)x}$ $(j = 0, \ldots, n)$. By Theorem I,

$$(I + \tilde{K})h = \sum c_j e_{\lambda + j} \qquad (j = 0, \ldots, n),$$

where $c_j = b_j \gamma_{\lambda+j}$ and

$$\|(I+\tilde{K})(wg-h)\|_u < \varepsilon C, \tag{2.3}$$

where C depends only on μ. The conclusion of the theorem now follows by combining (2.1), (2.2) and (2.3).

Remark 2.4. The Weierstrass theorem and Theorem II above show that for $\mu > Q^{1/2}$ and $\lambda = (\mu^2 + Q)^{1/2}$, the sets $S_1 = \{e^{-(\mu+j)x} : j = 0, 1, 2,...\}$ and $S_2 = \{e_{\lambda+j} : j = 0, 1,...\}$ both have a dense linear span in X_0, the real valued continuous functions which are $o(e^{-\mu x})$. However, these sets may be quite independent. For example, we can show the following: Suppose that in addition to the assumptions of Theorem II, q also satisfies $Q = 1$, $q(x) = 0$ on $\frac{1}{2} < x < \frac{3}{2}$ and $q(x) = 1$ for $x > 2$. Then only the zero function is common to the linear span of both S_1 and S_2. In fact, one can show that

$$\sum_{j=1}^{r} a_j e^{-\mu_j x} = \sum_{k=1}^{s} b_k e_{\lambda_k}(x), \tag{2.4}$$

where $\mu \leqslant \mu_1 < \mu_2 < \cdots$ and $\lambda \leqslant \lambda_1 < \lambda_2 < \cdots$, is possible only if all coefficients a_j and b_k are zero. If we differentiate (2.4) $2m$ times near $x = 1$ (where $q \equiv 0$), we obtain

$$\sum a_j(-\mu_j)^{2m} e^{-\mu_j x} = \sum b_k(\lambda_k)^{2m} e_{\lambda_k}(x) \tag{2.5}$$

for x ner 1 and for $m = 1, 2,...$. Since we may assume all coefficients a_j and b_k are $\neq 0$ and since $e_{\lambda_k}(1) \neq 0$ (see Lemma 1.1), we can conclude, in particular, that $\mu_1 = \lambda_1$, $\mu_2 = \lambda_2,...$. On the other hand, $e_{\lambda_k}(x)$ is a positive multiple of $\exp(-(1+\lambda_k^2)^{1/2}x)$ on $2 < x < \infty$. Therefore, by considering growth rates as $x \to +\infty$ for both sides of (2.4), we conclude that $\mu_1 = (1+\lambda_1^2)^{1/2}$; this and $\mu_1 = \lambda_1$ is impossible.

References

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